Weekly 1 Solutions

Overview: Below are some possible solutions to the exercises from Weekly 1. Keep in mind that there may be many ways to solve each problem. Your grade on Weekly 1 will be determined by the grader using the Weekly Assignment Rubric.

Solutions:

1. Exercise 9.1.15

Solution. a. This trace desribes how the volume V of 1 mole of an ideal gas is related to the temperature T if the pressure P is fixed at $P = 1000$ pascals. The trace is the curve described by the equation $z = V(1000, T) = 0.008314T$. Since this curve is a line, we can be very specific: the trace tells us that when the pressure is fixed at $P = 1000$ pascals, the volume V of 1 mole of a gas increases linearly with the temperature T at a rate of 0.008314 cubic meters per unit Kelvin.

b. This trace describes how the volume of 1 mole of a gas is related to the pressure P if the temperature is fixed at $T = 5$ Kelvin. The trace is the hyperbola described by the equation $z = 41.57/P$. This tells us that when the temperature is fixed at $T = 5$ Kelvin, the volume is inversely related to the pressure P. More specifically, if the temperature is fixed at $T = 5$ Kelvin, then the volume of 1 mole of a gas decreases as the pressure increases, and vice-versa

c. The level curve with $V = 0.5$ describes the relationship between the temperature T and the pressure P when the volume of 1 mole of a gas is fixed at $V = 0.5$ cubic meters. As long as the pressure is non-zero (meaning we are not in a perfect vacuum), the equation of the curve is $P = 16.628T$. This tells us that when the volume is fixed at $P = 0.5$ cubic meters, the pressure P increases linearly with the temperature T at a rate of 16.628 pascal per Kelvin.

d. You should have drawn a picture. I will just graph it: GeoGebra: Ideal Gas Law. Note that temperature are pressure are nonnegative by definition, therefore the domain of this function is $D = \{(P, T) : 0 \leq T,\ 0 \leq P\}$.

□

2. Exercise 9.3.11

Solution. a. The vectors are not perpendicular because $\langle 2, -1\rangle \cdot \mathbf{v} = 2(-2) + (-1)5 = -9$.

b. Suppose that $\mathbf{u} = \langle a, b\rangle$ is a unit vector so that $1 = |\mathbf{u}|^2 = a^2 + b^2$. Further assume that $\mathbf{u}$ is perpendicular to $\mathbf{v} = \langle -2, 5\rangle$ so that $0 = \mathbf{u} \cdot \mathbf{v} = -2a + 5b$. Now, we have a system of two equations that we can solve by substitution. We have $a = \frac{5}{2}b$ so that $1 = \frac{29}{4}b^2$. The second equation has precisely two solutions, $b = \pm\frac{2}{\sqrt{29}}$. So there are precisely two vectors that fit the bill, $\mathbf{u} = \pm\langle \frac{5}{\sqrt{29}}, \frac{2}{\sqrt{29}}\rangle$. This makes sense geometrically.

c. It is not, because the dot product is not zero.

d. The method to solve this part is similar to part (b). But in this case, there are infinitely many unit vectors perpendicular to $\mathbf{y}$. The set of terminal points of all possible vectors traces out a circle in $\mathbb{R}^3$ with radius equal to 1. Here's a visualization: GeoGebra: All Unit Vector Perp. to a Fixed Vector.

e. Suppose that $\mathbf{r} = \langle a, b, c\rangle$ is a unit vector perpendicular to both $\mathbf{y}$ and $\mathbf{z}$. Then we get a system of equations

$$\begin{cases} 1 &= a^2 + b^2 + c^2 \\ 0 &= 2a + b \\ 0 &= 3b - 2c. \end{cases}$$

By substituting the second two equations into the first, we get $1 = a^2 + 4a^2 + 9a^2$ so that $a = \pm\frac{1}{\sqrt{14}}$. So there are two unit vectors with the desired property, namely $\mathbf{r} = \pm\langle \frac{1}{\sqrt{14}}, -\frac{2}{\sqrt{14}}, -\frac{3}{\sqrt{14}}\rangle$. If you ignore the condition that $\mathbf{r}$ should have unit length, then the set of all terminal points of vectors perpendicular to both $\mathbf{y}$ and $\mathbf{z}$ is a line!
You could also have solved this part using the cross product.

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3. (16 points) Exercise 9.4.13.

Solution. (a) The displacement vector from P to Q is $Q - P = (-2, -4, 5)$. The displacement vector from P to R is $R - P = (1, 2, 1)$.

(b) The magnitude of the cross product of the vectors from (a) is equal to the area of the parallelogram determined by them. We have

$$\begin{vmatrix} i & j & k \\ -2 & -4 & 5 \\ 1 & 2 & 1 \end{vmatrix} = \begin{vmatrix} -4 & 5 \\ 2 & 1 \end{vmatrix} i - \begin{vmatrix} -2 & 5 \\ 1 & 1 \end{vmatrix} j + \begin{vmatrix} -2 & -4 \\ 1 & 2 \end{vmatrix} k = -14i + 7j.$$

The magnitude is $|(-14, 7, 0)| = \sqrt{14^2 + 7^2} = \sqrt{245} = 7\sqrt{5}$. The area of the triangle is $\frac{7\sqrt{5}}{2}$.

(c) The cross product $(-14, 7, 0)$ from (b) is already orthogonal to the plane containing the three points. To find a unit vector, divide it by its magnitude $7\sqrt{5}$.

(d) Use the dot product formula.

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